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GEODESICS AND VOLUMES IN REAL PROJECTIVE SPACES

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In 1951 P.M. Pu [11] proved the following result: Let P_2 be 2-dimensional real projective space, Γ the nontrivial free homotopy class of sectionally smooth ω: $[0, 1] \to P_2$, ω' the velocity vector of ω, h a Riemannian metric on P_2, $l_h = \inf\left\{\int_0^1 (h(\omega', \omega'))^{1/2} : \omega \in \Gamma\right\}$, and v_h the Riemannian volume of P_2 relative to h. Then $(l_h)^2/v_h \leq \frac{1}{2}\pi$, with equality if and only if h has constant sectional curvature.

Pu's method was based on the fact that h is a conformal deformation of a Riemannian metric on P_2 of constant sectional curvature, that he could therefore average the metric over the group of isometries of P_2 with the Riemannian metric of constant sectional curvature, and then show than l_h increases and v_h decreases.

In this note we will consider two examples related to the appropriate (yet unsolved) generalization of Pu's result to higher dimensions. For convenience, we formulate our problem as a conjecture:

Pu's conjecture. Let P_n denote n-dimensional real projective space, Γ the nontrivial free homotopy class of continuous sectionally smooth ω: $[0, 1] \to P_n$, g the Riemannian metric of constant sectional curvature 1 on P_n, and h any Riemannian metric on P_n. Set $l_h = \inf\left\{\int_0^1 (h(\omega', \omega'))^{1/2} : \omega \in \Gamma\right\}$, where ω' is the velocity vector of ω, and v_h to be the Riemannian volume of P_n relative to h. Then

$$(l_h)^n/v_h \leq (l_g)^n/v_g$$

with equality if and only if h has constant sectional curvature.

One easily sees that

$$(l_g)^n/v_g = \begin{cases} k!\pi^k, & n = 2k+1 , \\ (\pi/2)^k(2k-1)(2k-3)\cdots 3.1 , & n = 2k . \end{cases}$$

In § 1 we consider a 1-parameter family of Riemannian homogeneous metrics

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on $P_{2k+1}, k \geq 1$, which are of strictly positive sectional curvature. For each member h of the family we explicitly calculate l_h, v_h, and show $(l_h)^{2k+1}/v_h \leq k!\pi^k$, with equality if and only if $k = 1$, and h has constant sectional curvature (for $k > 1$ none of the Riemannian metrics considered have constant sectional curvature). This class of Riemannian homogeneous metrics on P_{2k+1} was first discovered by *M*. Berger [1] and subsequently investigated in [6], where the reader will find the details necessary for our discussion.

In § 2 we generalize Pu's method of averaging Riemannian metrics on a normal Riemannian homogeneous space over the group of isometries of the space. In § 3, however, we show the limitations of Pu's method in higher dimensions, viz, for the complete collection of Poincaré metrics [10] on P_3 induced by Riemannian metrics on the 2-sphere S_2 (where P_3 is identified with the unit tangent bundle of S_2), averaging the Poincaré metric *increases* the volume instead of decreasing it (the length of the minimal geodesic increases as before). Furthermore, one can make the ratio of the volume of the averaged metric to the volume of the original one larger than any given positive constant, so that Pu's method does not even supply *any* upper bound for the function $(l_h)^n/v_h$ on this restricted collection of Riemannian metrics on P_3. We refer the reader to [12] for details concerning these metrics. In a forthcoming paper we will present some positive results concerning Pu's problem for Poincaré metrics on P_3.

Pu's original approach does work in higher dimensions for conformal deformations of Riemannian homogeneous metrics, as Pu himself noted [11, p. 62] and the approach is successful for the linearized version of the conjecture— cf. [4], [7] for details. In these papers and in [3] one will also find discussions of the relation of Pu's conjecture to Blaschke's conjecture on *wiedersehnsraume*. Finally, we remark that in [3], [4] Pu's conjecture is discussed for higher dimensional subspaces of all the projective spaces.

1. Extremal lengths in odd-dimensional real projective space

Let E_{rs} denote the $(n+1) \times (n+1)$ matrix which has a 1 in the r-th row and s-th column and 0 elsewhere, and set: $A_{rs} = \sqrt{-1}(E_{rr} - E_{ss})$, $B_{rs} = E_{rs} - E_{sr}$, $C_{rs} = \sqrt{-1}(E_{rs} + E_{sr})$, where $r, s = 1, \cdots, n+1$. Also, set

$$\alpha_j = \{\tfrac{1}{2}j(j+1)\}^{1/2}\,, \qquad S_j = \sum_{k=1}^{j} kA_{k,k+1}/\alpha_j\,,$$

where $j = 1, \cdots, n$. If $\mathfrak{a}_n$ denotes the Lie algebra of $SU(n+1)$, the special unitary group acting on $(n+1)$-dimensional complex number space, $n \geq 1$, with the inner product $\langle x, y\rangle = -\frac{1}{2}$ trace (xy), then an orthonormal basis of $\mathfrak{a}_{n-1}$ naturally imbedded in $\mathfrak{a}_n$ is given by $\{S_1, \cdots, S_{n-1};\ B_{jk}, C_{jk}\colon 1 \leq j < k \leq n\}$. Let $\mathfrak{g}$ be the direct orthogonal sum $\mathfrak{a}_n \oplus R$, R = real numbers, D be a basis element of R of unit length, and $[\mathfrak{a}_n, R] = 0$, where [,] denotes Lie multiplication. Thus $\mathfrak{g}$ is a Lie algebra.

Fix a real number $\alpha, 0 < \alpha < \pi$; let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ spanned by $\{S_1, \cdots, S_{n-1}, \cos\alpha \cdot S_n + \sin\alpha \cdot D;\ B_{jk}, C_{jk}\colon 1 \le j < k \le n\}$, $G = \exp \mathfrak{g}$, $\hat{H} = \exp \mathfrak{h}$, where exp denotes the exponential map of the Lie algebra to the Lie group it generates, and set $\hat{M} = G/\hat{H}$, the resulting homogeneous space. The orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, which we denote by $\mathfrak{m}$, has an orthonormal basis given by $\{\sin\alpha \cdot S_n - \cos\alpha \cdot D;\ B_{j,n+1}, C_{j,n+1}\colon j = 1, \cdots, n\}$. Thus the dimension of $\hat{M}$ is $2n + 1$. If $\hat{\pi}\colon G \to \hat{M}$ denotes the natural projection $\hat{o} = \hat{\pi}(\hat{H})$, then the tangent space $\hat{M}_{\hat{o}}$ to $\hat{M}$ at $\hat{o}$ is identified with $\mathfrak{m}$ from which $\hat{M}$ obtains a natural Riemannian homogeneous metric. The linear action of $\hat{H}$ on $\hat{M}_{\hat{o}}$ is given by $\mathrm{Ad}(\hat{H})$ acting on $\mathfrak{m}$; also

$$[\mathfrak{a}_{n-1}, S_n] = 0\ , \tag{1}$$

$$\dim\, [\mathfrak{a}_{n-1}, B_{1,n+1}] = 2n - 1\ . \tag{2}$$

We therefore have

Proposition 1. $\mathrm{Ad}(\hat{H})$ *leaves* $\sin\alpha \cdot S_n - \cos\alpha \cdot D$ *invariant, and acts transitively on the unit sphere of the orthogonal complement of* $\sin\alpha \cdot S_n - \cos\alpha \cdot D$ *in* $\mathfrak{m}$.

We note that if $\hat{\mathrm{Exp}}_{\hat{o}}\colon \mathfrak{m} \to \hat{M}$ denotes the Riemannian exponential map, then for any $x \in \mathfrak{m}$ we have $\hat{\mathrm{Exp}}_{\hat{o}}\, x = \hat{\pi}(\exp x)$. Also one easily sees that $\exp tS_n$ is given by

$$\exp tS_n = \begin{bmatrix} e^{(\sqrt{-1}/\alpha_n)t} & & & \\ & \ddots & & \\ & & e^{(\sqrt{-1}/\alpha_n)t} & \\ & & & e^{-(\sqrt{-1}n/\alpha_n)t} \end{bmatrix},$$

and therefore $\exp tS_n \in SU(n)$ if and only if t is an integral multiple of $2\pi\alpha_n/n$. Since $\mathfrak{a}_{n-1}, R\cdot S_n, R\cdot D$ all commute, and any geodesic loop in a normal Riemannian homogeneous space is simply closed [9, Theorem 3], we have (see Fig. 1) that the *$\hat{M}$-geodesic generated by* $e_0 = \sin\alpha \cdot S_n - \cos\alpha \cdot D$ *is simply*

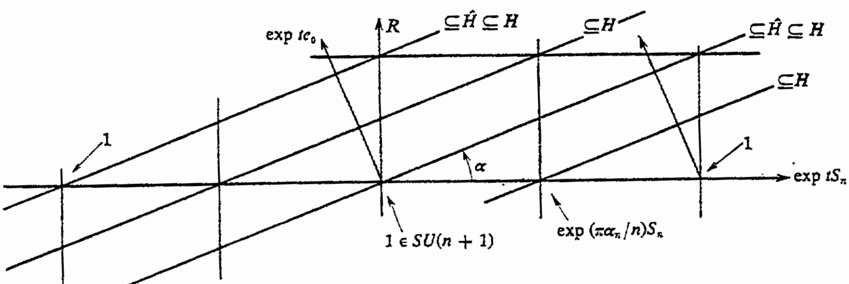

Fig. 1

closed and of length $(2\pi\alpha_n/n)\sin\alpha$.

Let H^* be the Lie group generated by $SU(n)$ and $\exp(\pi\alpha_n/n)S_n$, H be the Lie group generated by $\hat{H}$ and $\exp(\pi\alpha_n/n)\sin\alpha\cdot e_o$, $M = G/H$, and $\pi: G \to M$ be the natural projection. Then the natural map $p: \hat{M} \to M$ satisfying $p \circ \hat{\pi} = \pi$ is a Riemannian covering with fibre $H/\hat{H}$ homeomorphic to Z_2. By using Lemma 2 (to be stated below) one easily checks that $\hat{M}$ is homeomorphic to $SU(n+1)/SU(n)$ which is homeomorphic to a sphere, and that M is homeomorphic to $U(n+1)/\{U(n), -1\}$ (where $U(n+1)$ is the unitary group acting on $n+1$ complex variables, and $\{U(n), -1\}$ is the group generated by $U(n)$ naturally imbedded in $U(n+1)$, and minus the identity $\in U(n+1)$) which is homeomorphic to $(2n+1)$-dimensional real projective space.

Lemma 2 [1, *Proposition* 3.2]. *Let G be a compact Lie group, and K, H, L closed subgroups of G such that* (1) $K \supseteq L$, (2) $L = K \cap H$, (3) *the Lie algebra $\mathfrak{l}$ of L has a complementary subspace in the Lie algebra $\mathfrak{k}$ of K which is also a complementary subspace of H in G. Then G/H is homeomorphic to K/L.*

We now let Γ denote the free nontrivial homotopy class of continuous sectionally smooth $\omega: [0,1] \to M$, $l(\omega)$ the length of any given $\omega \in \Gamma$, and l_α the $\inf\{l(\omega): \omega \in \Gamma\}$.

Proposition 3. $l_\alpha = (\pi\alpha_n/n)\sin\alpha$.

Proof. By the homogeneity of M it suffices to consider $\omega \in \Gamma$ satisfying $\omega(0) = \pi(H)$. Furthermore, it is well-known that Γ has a simple closed geodesic whose length is equal to l_α. By Proposition 1 it remains to look at all geodesics γ_θ, $0 \le \theta \le \frac{1}{2}\pi$, emanating from $o = \pi(H)$ with initial velocity vectors $\gamma_\theta'(0) = \cos\theta\cdot e_o + \sin\theta\cdot B_{1,n+1}$ (it is clear that $\mathfrak{m}$ is identified with the tangent space to M at o). Now $\gamma_0 \in \Gamma$ and is simply closed of length $(\pi\alpha_n/n)\sin\alpha$. Therefore let $0 < \theta \le \frac{1}{2}\pi$ and set

$$\beta = (2(n+1)/n)^{1/2}\sin\alpha\,, \qquad \sigma(\theta) = (4\sin^2\theta + \beta^2\cos^2\theta)^{1/2}.$$

We claim that if γ_θ is simply closed, then its length is an integral multiple of $2\pi/\sigma(\theta)$. Indeed, by Proposition 1, $\mathrm{Ad}(H)$ acts on $\gamma_\theta'(0)$ nontrivially and therefore induces a Jacobi field along γ_θ which vanishes for $t = 0$ and (at least) for all integral multiples of the length of γ_θ; but by Theorem 1 of [6] the zeroes of all Jacobi fields induced by H vanish precisely at the integral multiples of $2\pi/\sigma(\theta)$ and nowhere else. One now checks easily that $(\pi\alpha_n/n)\sin\alpha \le 2\pi/\sigma(\theta)$ for all θ, and the proposition is proven.

Theorem 4. *Let v_α denote the volume of M. Then*

$$(3) \qquad (l_\alpha)^{2n+1}/v_\alpha = n!\,\pi^n(\alpha_n/n)^{2n}\sin^{2n}\alpha \le n!\,\pi^n$$

with equality if and only if $n = 1$, $\alpha = \frac{1}{2}\pi$, i.e., M is 3-dimensional real projective space of constant sectional curvature 1.

Proof. Since $(\alpha_n/n) \le 1$ for all $n \ge 1$, with equality if and only if $n = 1$, to complete the proof it suffices to calculate the volume of M. We quote the lemma on "integration over the fibers" from [5, p. 16] for present and subsequent use (§ 2).

Lemma 5. *Let $\hat{M}, M$ be Riemannian manifolds with metric tensors $\hat{g}, g$ respectively, $\dim \hat{M} > \dim M$, and $\pi\colon \hat{M} \to M$ a Riemannian submersion [5, p. 16]. Denote the induced Riemannian measures by $dv_{\hat{g}}, dv_g$ respectively, and for any $p \in M$ let dv_p denote the induced Riemannian measure on the fiber $\pi^{-1}(p)$. Furthermore, let $\hat{f}\colon \hat{M} \to R$ be continuous with compact support, and $f\colon M \to R$ be the function on M defined by $f(p) = \int_{\pi^{-1}(p)} \hat{f}|_{\pi^{-1}(p)} \cdot dv_p$. Then f is continuous with compact support and one has*

$$\int_{\hat{M}} \hat{f} \cdot dv_{\hat{g}} = \int_M f \cdot dv_g. \tag{4}$$

We return to v_α which is $\frac{1}{2}$ vol $\hat{M}$. Lemma 5 and Fig. 1 combine to imply vol $\hat{M} = \sin\alpha \cdot \text{vol}\, SU(n+1)/SU(n)$. Now the natural map of

$$SU(n+1)/SU(n) \to SU(n+1)/S(U(n) \times U(1))$$

is a Riemannian submersion actually, a fibration) with fiber a circle group $S(U(n) \times U(1))/SU(n)$, i.e., $\{\exp tS_n\}$ with length $2\pi\alpha_n/n$. (Since our metrics are chosen such that every element in $SU(n+1)$ acts as an isometry, all the fibers are of the same length). Also, $SU(n+1)/S(U(n) \times U(1))$ is isometric to complex projective space CP_n of $2n$ real dimensions with the standard Fubini-Study metric, and assuming curvature values between 1 and 4. The volume of CP_n is therefore $\pi^n/n!$ (cf. [5, pp. 18, 112]) and the volume of $SU(n+1)/SU(n)$ is $2\pi^{n+1}\alpha_n/(n(n!))$ by Lemma 5, and the theorem follows easily.

2. Averaging of symmetric 2-tensors on Riemannian homogeneous spaces

We start with

Lemma 6. *Let H be a closed subgroup of the orthogonal group $O(n)$, and assume that H acts irreducibly on R^n with the standard inner product $\langle\ ,\ \rangle$. Let $B\colon R^n \times R^n \to R$ be a symmetric bilinear form on R^n, dv_H be a bi-invariant measure on H, and $v_H = \int_H dv_H$. Define the bilinear form $b\colon R^n \times R^n \to R$ by*

$$b(x, y) = \frac{1}{v_H} \int_H B(h \cdot x, h \cdot y) dv_H.$$

Then for all $x, y \in R^n$ we have

$$b(x, y) = \frac{1}{n}(\text{trace } B)\langle x, y\rangle .$$

Proof. Clearly, there exists a $\kappa \in R$ such that $b(x, y) = \kappa \cdot \langle x, y\rangle$ for all $x, y \in R^n$, so it remains to calculate κ. Pick an orthonormal basis $\{e_1, \cdots, e_n\}$ of R^n, and set $B_{jk} = B(e_j, e_k)$. For $h \in H$, let $a_{rs}(h)$ denote the matrix associated to h and the basis $\{e_1, \cdots, e_n\}$. Then

$$n\kappa = \sum_{k=1}^{n} b(e_k, e_k) = \sum_{k=1}^{n} \frac{1}{v_H} \int_H B(h \cdot e_k, h \cdot e_k) dv_H$$

$$= \frac{1}{v_H} \int_H \sum_{k,j,l=1}^{n} a_{jk}(h) a_{lk}(h) B_{lj} dv_H$$

$$= \frac{1}{v_H} \int_H \sum_{j=1}^{n} B_{jj} dv_H = \text{trace } B.$$

Theorem 7. *Let K be a compact connected Lie group with bi-invariant Riemannian metric and induced bi-invariant measure dv_K, and $v_K = \int_K dv_K$. Let L be a closed (and hence compact) subgroup, and K/L the resulting homogeneous space with naturally induced Riemannian homogeneous metric g. Let dv_g denote the induced Riemannian measure on $K/L, v_g = \int_{K/L} dv_g$, and $n = \dim(K/L)$. If t is any symmetric 2-tensor field on K/L, $p \in K/L$, and $x, y \in (K/L)_p$ the tangent space to K/L at p, then the tensor field $\tilde{t}$ on K/L defined by*

$$\tilde{t}(x, y) = \frac{1}{v_K} \int_K t(k \cdot x, k \cdot y) dv_K$$

is an invariant symmetric 2-tensor on K/L, where $k \cdot x$ denotes the linear action of $k \in K$ on tangent vectors to K/L. If L acts on K/L irreducibly, then

$$\tilde{t} = \left\{ \frac{1}{nv_g} \int_{K/L} \tau dv_g \right\} \cdot g ,$$

where τ is the trace of t relative to g.

Proof. Using the invariance of dv_K, one easily obtains the invariance of $\tilde{t}$ under the action of K. Now let $\pi: K \to K/L$ denote the standard projection, $o = \pi(L)$, $x, y \in (K/L)_o$ (clearly, by homogeneity and invariance it suffices to consider this case), and assume L acts irreducibly on K/L. Let dv_L denote the induced bi-invariant measure on L, $v_L = \int_L dv_L$, and for $k \in K$, $p = \pi(k) = k \cdot L$ let L_p denote the fixed point group of p, i.e., $L_p = kLk^{-1}$. In our subsequent integration formulas we let l range over L, and l_p over L_p. Then

$$\begin{aligned}\bar{t}(x, y) &= \frac{1}{v_K}\int_K t(k\cdot x, k\cdot y)dv_K \\ &= \frac{1}{v_K}\int_{K/L}\left\{\int_{\pi^{-1}(p)} t(kl\cdot x, kl\cdot y)dv_p\right\}dv_g \\ &= \frac{1}{v_K}\int_{K/L}\left\{\int_{L_p} t(l_pk\cdot x, l_pk\cdot y)dv_{L_p}\right\}dv_g \\ &= \frac{1}{v_K}\int_{K/L}\frac{\tau(p)}{n}g(k\cdot x, k\cdot y)v_{L_p}dv_g \\ &= \left\{\frac{v_L}{v_K}\int_{K/L}\frac{\tau}{n}dv_g\right\}\cdot g(x, y) \\ &= \left\{\frac{1}{nv_g}\int_{K/L}\tau dv_g\right\}\cdot g(x, y)\ .\end{aligned}$$

To go from the first line to the second one uses Lemma 5 (v_p is the Riemannian measure of the coset $\pi^{-1}(p)$), from the second to the third is obvious: (dv_{L_p} is the bi-invariant measure on L_p), from the third to the fourth one uses Lemma 6, from the fourth to the fifth one uses the invariance of the metrics and their induced measures, and the final line is obtained using the invariance of the measures and Lemma 5 as in § 1.

Corollary 8. *Let* det $\bar{t}$ *denote the determinant of* $\bar{t}$ *relative to* g. *Then*

$$\int_{K/L}(\det \bar{t})^{1/2}v_g = \left\{\frac{1}{n}\int_{K/L}\tau dv_g\right\}^{n/2}\cdot v_g^{\,1-n/2}\ .$$

We note that if Γ is a nontrivial free homotopy class of continuously sectionally smooth ω: $[0, 1] \to K/L$, and $E_t = \inf\left\{\int_0^1 t(\omega', \omega') : \omega \in \Gamma\right\}$ and similarly for $E_{\bar{t}}$, then $E_{\bar{t}} \geq E_t$. Indeed, for every $\omega \in \Gamma$, $k\cdot\omega \in \Gamma$ by the connectivity of K, which implies

$$\int_0^1 \bar{t}(\omega', \omega') = \int_0^1 \frac{1}{v_K}\int_K t(k\cdot\omega', k\cdot\omega')dv_K$$

$$= \frac{1}{v_K}\int_K \left\{\int_0^1 t(k\cdot\omega', k\cdot\omega')\right\} dv_K \geq \frac{1}{v_K}\int_K E_t dv_K = E_t .$$

If t is positive definite, then $l_t = \inf\left\{\int_0^1 (t(\omega', \omega'))^{1/2} : \omega \in \Gamma\right\} = E_t^{1/2}$, and similarly for $\tilde{t}$ which implies $l_{\tilde{t}} \geq l_t$. We also note that if t is a conformal deformation of g, then by using Hölder's inequality one easily shows that $\int_{K/L} (\det \tilde{t})^{1/2} dv_g \leq \int_{K/L} (\det t)^{1/2} dv_g$; so if t is a positive conformal deformation of g, and $v_t, v_{\tilde{t}}$ denote the appropriate volumes, $l_t^n/v_t \leq l_{\tilde{t}}^n/v_{\tilde{t}} = l_g^n/v_g$, which is Pu's original result. We now turn to a class of Riemannian metrics $\{h\}$ on 3-dimensional real projective space P_3 for which $v_{\tilde{h}} \geq v_h$.

3. Poincaré metrics on 3-dimensional real projective space

We now let S_2 be the standard 2-sphere. Then for any Riemannian metric $\hat{g}$ on S_2 the unit tangent bundle of the metric is homeomorphic to P_3. (Indeed, for different metrics on S_2 the unit tangent bundles are homeomorphic, and for constant sectional curvature 1 the unit tangent bundle is explicitly seen to be the special orthogonal group $SO(3)$ acting on R^3. But $SO(3)$ is known to be homeomorphic to P_3 (e.g. cf. [13, p. 115])). To construct the induced Riemannian metric g on P_3 we first construct in the standard manner (cf. [8, Chap. III, IV] for details) the 3 global linearly independent differential 1-forms on P_3 (viewed as the oriented frame bundle of S_2): $\omega_1, \omega_2, \omega_{21}$ where ω_1, ω_2 are the canonical forms of the bundle, and ω_{21} is the connection form on P_3 of $\hat{g}$. The forms $\omega_1, \omega_2, \omega_{21}$ then satisfy the Cartan structure equations

$$d\omega_1 = \omega_2 \wedge \omega_{21} , \qquad d\omega_2 = -\omega_1 \wedge \omega_{21} , \qquad d\omega_{21} = K\omega_1 \wedge \omega_2 ,$$

where K is the Gaussian curvature of $\hat{g}$ (actually we should write $K \circ \pi$). The induced Riemannian metric on P_3 is then defined by $(ds_g)^2 = (\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2$, i.e., at each point the global linear frame on P_3 dual to the basis of 1-forms $\{\omega_1, \omega_2, \omega_{21}\}$ is declared to be orthonormal. One checks that this metric is the same as the construction given in local coordinates in [12].

We henceforth let $\hat{g}$ be the metric on S_2 of constant sectional curvature 1 with associated forms $\omega_1, \omega_2, \omega_{21}$ on P_3, and for any other Riemannian metric $\hat{h}$ on S_2 we denote the associated forms P_3 by $\omega_1{}^h, \omega_2{}^h, \omega_{21}{}^h$. First we note that g has constant sectional curvature 1. Second, for any given metric $\hat{h}$ on S_2 there exists $\sigma: S_2 \to R$ such that $\hat{h} = e^{2\sigma}\hat{g}$ which implies that $d\sigma$ (lifted to P_3) is of the form $d\sigma = \sigma_1\omega_1 + \sigma_2\omega_2$, and $\omega_1{}^h = e^\sigma\omega_1, \omega_2{}^h = e^\sigma\omega_2, \omega_{21}{}^h = \sigma_2\omega_1 - \sigma_1\omega_2 + \omega_{21}$. Direct calculation then shows

$$\operatorname{trace}_g h = 2e^{2\sigma} + 1 + \|\operatorname{grad}\sigma\|^2 , \qquad \det{}_g h = e^{4\sigma} ,$$

where $\| \operatorname{grad} \sigma \|^2 = (\sigma_1^2 + \sigma_2^2)$ is the length squared of the gradient of σ on S_2 in the metric $\hat{g}$.

The Riemannian metric g of constant sectional curvature 1 on P_3 can be realized naturally from the Riemannian homogeneous space $SO(4)/\{SO(3), -1\}$, where for any positive integer n, $SO(n)$ denotes the special orthogonal group acting on R^n, and $\{SO(3), -1\}$ denotes the group generated by $SO(3)$ naturally imbedded in $SO(4)$ and minus the identity map of R^4. Let $\tilde{h}$ denote the Riemannian metric on P_3 obtained by averaging h over $SO(4)$ as in Theorem 7, and $v_h, v_{\tilde{h}}$ the respective volumes of P_3. Then by Lemma 5 and Corollary 8 we have

$$v_h = 2\pi \int_{S_2} e^{2\sigma} dv_g \,,$$

$$v_{\tilde{h}} = \frac{1}{\sqrt{8\pi}} \left\{ \frac{1}{3} \left(2v_h + 8\pi^2 + 2\pi \int_{S_2} \| \operatorname{grad} \sigma \|^2 dv_g \right) \right\}^{3/2} .$$

Let $\alpha = \dfrac{v_h}{8\pi^2} = \dfrac{1}{4\pi} \displaystyle\int_{S_2} e^{2\sigma} dv_{\hat{g}}$. Then

$$\begin{aligned} (v_{\tilde{h}})^2 - (v_h)^2 &\geq (8\pi^2)^2 \{(2\alpha + 1)^3 - 27\alpha^2\}/27 \\ &= (8\pi^2)^2 (\alpha - 1)^2 (8\alpha + 1)/27 \end{aligned}$$

with equality if and only if $\sigma \equiv 0$. Thus $v_{\tilde{h}} \geq v_h$ with equality if and only if $h = \hat{g}$. Furthermore, if κ is any positive constant and σ any constant $> \frac{1}{2} ln(27\kappa/8)$, then $(v_{\tilde{h}})^2 - \kappa (v_h)^2 > 0$.

The last comment in the above paragraph reflects the fact that a Rimannian metric of constant sectional curvature $\kappa \neq 1$ on S_2 does not induce a metric of constant sectional curvature on P_3. If we change our class of metrics on P_3 by replacing $\omega_{21}{}^h$ written above with $\omega_{21}{}^{h^*} = e^{2\sigma}(\sigma_2\omega_1 - \sigma_1\omega_2 + \omega_{21})$, then metrics of constant sectional curvature on S_2 induce metrics of constant sectional curvature on P_3. Furthermore, $(l_{h^*})^3/v_{h^*} \leq \pi$ with equality if and only if σ is constant. Indeed $v_{h^*} = 2\pi \displaystyle\int_{S_2} e^{3\sigma} dv_{\hat{g}}$; each of the fibers is nonhomotopic to zero [13, p. 115] and of length $2\pi e^\sigma$, which implies $l_{h^*} \leq 2\pi \cdot \min_{S_2} e^\sigma \leq \frac{1}{2} \displaystyle\int_{S_2} e^\sigma dv_{\hat{g}}$. Hölder's inequality then implies $(l_{h^*})^3 \leq 2\pi^2 \displaystyle\int_{S_2} e^{3\sigma} dv_{\hat{g}}$ and the result follows. In light of N. Kuiper's remark [2, p. 309] the Poincaré metrics are of interest with regard to Pu's conjecture and, as mentioned, we will consider them in a future publication.

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